Learning Targets:

- Know and apply the Rational Root Theorem and Descartes' Rule of Signs.
- Know and apply the Remainder Theorem and the Factor Theorem.

SUGGESTED LEARNING STRATEGIES: Shared Reading, Vocabulary Organizer, Marking the Text, Note Taking

Some polynomial functions, such as \( f(x) = x^3 - 2x^2 - 5x + 6 \), are not factorable using the tools that you have. However, it is still possible to graph these functions without a calculator. The following tools will be helpful.

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Rational Root Theorem

If a polynomial function \( f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \), \( a_n \neq 0 \), has integer coefficients, then every rational root of \( f(x) = 0 \) has the form \( \frac{p}{q} \), where \( p \) is a factor of \( a_0 \), and \( q \) is a factor of \( a_n \).

The Rational Root Theorem determines the possible rational roots of a polynomial.

1. Consider the quadratic equation \( 2x^2 + 9x - 3 = 0 \).
   a. Make a list of the only possible rational roots to this equation.

   \[
   \begin{align*}
   p &= \pm 3, \pm 1 \\
   q &= \pm 3, \pm 1 \\
   \end{align*}
   \]

   Possible roots: \( \pm 3, \pm \frac{3}{2}, \pm 1, \pm \frac{1}{2} \).

   b. Reason abstractly. Explain why you think these are the only possible rational roots.

   \( p \)'s are the factors of -3

   \( q \)'s are the factors of 2

   Rational Root Theorem

   c. Does your list of rational roots satisfy the equation?

   \[
   x = \frac{-9 \pm \sqrt{9 - 4(-3)} \pm \sqrt{81 + 24}}{2(3)}
   \]

   \[
   x = \frac{-9 \pm \sqrt{9}}{4}
   \]

   \[
   x = \frac{-9 \pm 3}{4}
   \]

   \[
   x = \frac{-9 \pm \sqrt{105}}{14}
   \]
2. What can you conclude from Item 1 part c?

Rational Root Theorem gives the possible rational roots, but not irrational.

3. Reason quantitatively. Verify your conclusion in Item 1 part c by finding the roots of the equation in Item 1 using the Quadratic Formula. Show your work.

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Example A
Find all the possible rational roots of \( f(x) = x^3 - 2x^2 - 5x + 6 \).

Step 1: Find the factors \( q \) of the leading coefficient 1 and the factors \( p \) of the constant term 6.

\( q \) could equal \( \pm 1 \)
\( p \) could equal \( \pm 1, \pm 2, \pm 3, \pm 6 \)

Step 2: Write all combinations of \( \frac{p}{q} \). Then simplify.

\( \pm 1, \pm 2, \pm 3, \pm 6 \)

Solution: \( \pm 1, \pm 2, \pm 3, \pm 6 \)

Try These A
Find all the possible rational roots of \( f(x) = 2x^3 + 7x^2 + 2x - 3 \).

\[ \frac{p}{q} = \pm \frac{3}{1}, \pm \frac{2}{1}, \pm \frac{1}{1} \]

The Rational Root Theorem can yield a large number of possible roots. To help eliminate some possibilities, you can use Descartes’ Rule of Signs. While Descartes’ rule does not tell you the value of the roots, it does tell you the maximum number of positive and negative real roots.

Descartes’ Rule of Signs

If \( f(x) \) is a polynomial function with real coefficients and a nonzero constant term arranged in descending powers of the variable, then

- The number of positive real roots of \( f(x) = 0 \) equals the number of variations in sign of the terms of \( f(x) \), or is less than this number by an even integer.
- The number of negative real roots of \( f(-x) = 0 \) equals the number of variations in sign of the terms of \( f(-x) \), or is less than this number by an even integer.
Lesson 18-2
Finding the Roots of a Polynomial Function

Example B
Find the number of positive and negative roots of \( f(x) = x^3 - 2x^2 - 5x + 6 \).

**Step 1:** Determine the sign changes in \( f(x) = x^3 - 2x^2 - 5x + 6 \)
- There are two sign changes:
  - one between the first and second terms when the sign goes from positive to negative
  - one between the third and fourth terms when the sign goes from negative to positive

So, there are either two or zero positive real roots.

**Step 2:** Determine the sign changes in \( f(-x) = (-x)^3 - 2(-x)^2 - 5(-x) + 6 \)
- There is one sign change:
  - between the second and third terms when the sign goes from negative to positive

So, there is one negative real root.

**Solution:** There are either two or zero positive real roots and one negative real root.

Try These B
Find the number of positive and negative roots of \( f(x) = 2x^3 + 7x^2 + 2x - 3 \).

\[
\begin{align*}
\text{One positive} & \quad \text{Two or zero negative}
\end{align*}
\]

Check Your Understanding

4. Determine all the possible rational roots of \( f(x) = 2x^3 - 2x^2 - 4x + 5 \).
5. Determine the possible number of positive and negative real zeros for \( h(x) = x^3 - 4x^2 + x + 5 \).
6. The function \( f(x) = x^3 + x^2 + x + 1 \) has only one possible rational root. What is it? Explain your reasoning.
7. **Construct viable arguments.** Explain the circumstances under which the only possible rational roots of a polynomial are integers.

You have found all the possible rational roots and the number of positive and negative real roots for the function \( f(x) = x^3 - 2x^2 - 5x + 6 \). The next two theorems will help you find the zeros of the function. The Remainder Theorem tells if a possible root is actually a zero or just another point on the graph of the polynomial. The Factor Theorem gives another way to test if a possible root is a zero.
ACTIVITY 18
continued

Lesson 18-2
Finding the Roots of a Polynomial Function

Remainder Theorem
If a polynomial \( P(x) \) is divided by \( (x - k) \), where \( k \) is a constant, then the remainder \( r \) is \( P(k) \).

Factor Theorem
A polynomial \( P(x) \) has a factor \( (x - k) \) if and only if \( P(k) = 0 \).

Example C
Use synthetic division to find the zeros and factor \( f(x) = x^3 - 2x^2 - 5x + 6 \).

From Examples A and B, you know the possible rational zeros are \( \pm 1, \pm 2, \pm 3, \pm 6 \). You also know that the polynomial has either two or zero positive real roots and one negative real root. Now it is time to check each of the possible rational roots to determine if they are zeros of the function.

Step 1: Divide \( (x^3 - 2x^2 - 5x + 6) \) by \( (x + 1) \).

\[
\begin{array}{c|cccc}
-1 & 1 & -2 & -5 & 6 \\
 & & 1 & 3 & 2 \\
\hline
 & 1 & -3 & -2 & 8 \\
\end{array}
\]
So, you have found a point, \((-1, 8)\).

Step 2: Continue this process, finding either points on the polynomial and/or zeros for each of the possible roots.

Divide \( (x^3 - 2x^2 - 5x + 6) \) by \( (x - 1) \).

\[
\begin{array}{c|cccc}
1 & 1 & -2 & -5 & 6 \\
 & 1 & -1 & -6 & \\
\hline
 & 1 & -1 & -6 & 0 \\
\end{array}
\]
So, you have found a zero, \((1, 0)\), and a factor, \( f(x) = (x - 1)(x^2 - x - 6) \).

Step 3: As soon as you have a quadratic factor remaining after the division process, you can factor the quadratic factor by inspection, if possible, or use the Quadratic Formula.

Solution: \( f(x) = (x - 1)(x + 2)(x - 3) \); The real zeros are 1, -2, and 3.

Try These C
Use synthetic division and what you know from Try These A and B to find the zeros and factor \( f(x) = 2x^3 + 7x^2 - 2x - 3 \).

Possible zeros: \( \pm \frac{1}{2}, \pm \frac{3}{2}, \pm 1, \pm 3 \)

\[
f(x) = \frac{(x+3)(2x-1)(x-1)}{2}
\]

Check Your Understanding

8. One of the possible rational roots of \( f(x) = x^3 - 2x^2 - 4x + 5 \) is 5. If you divide \( x^3 - 2x^2 - 4x + 5 \) by \( x - 5 \), the remainder is 60. What information does this give you about the graph of \( f(x) \)?

9. If a polynomial \( P(x) \) is divided by \( (x - k) \) and the remainder is 0, what does this tell you about the value \( k \)?
Lesson 18-2
Finding the Roots of a Polynomial Function

Using the Factor Theorem, follow a similar process to find the real zeros.

Example D
Use the Factor Theorem to find the real zeros of \( f(x) = x^3 - 2x^2 - 5x + 6 \).
Again, you know the possible rational roots are \( \pm 1, \pm 2, \pm 3, \pm 6 \).

Step 1: Test \( (x + 1) \): \( f(-1) = (-1)^3 - 2(-1)^2 - 5(-1) + 6 = 8 \)
So, you have a point, \((-1, 8)\).

Step 2: Test \( (x - 1) \): \( f(1) = (1)^3 - 2(1)^2 - 5(1) + 6 = 0 \)
So, you have a zero at \( x = 1 \).

Step 3: Test \( (x - 2) \): \( f(2) = (2)^3 - 2(2)^2 - 5(2) + 6 = -4 \)
So, you have a point, \((2, -4)\).

Step 4: Continue to test rational zeros or use division to simplify the polynomial and factor or use the Quadratic Formula to find the real zeros.

Solution: The real zeros are 1, -2, and 3.

Try These D
Use the Factor Theorem and what you know from Try These A and B to find the real zeros of \( f(x) = 2x^3 + 7x^2 + 2x - 3 \).

Similar to Synthetic

Example E
Graph \( f(x) = x^3 - 2x^2 - 5x + 6 \) using the information you found in Examples A–D. Also include the y-intercept and what you know about the end behavior of the function.
Try These E
Graph \( f(x) = 2x^3 + 7x^2 + 2x - 3 \) using the information from Try These A–D. Include a scale on both axes.

Check Your Understanding

10. The possible rational roots of \( g(x) = 2x^4 + 5x^3 - x^2 + 5x - 1 \) are \( ±\frac{1}{2} \) and \( ±1 \). List the possible factors of \( g(x) \).

11. For the function \( f(x) = x^3 - 2x^2 - 4x + 5 \), \( f(-1) = 6 \). Is \( x + 1 \) a factor of \( f(x) \)? Explain your reasoning.

12. For the function \( p(x) = x^3 - 2x^2 - 4x + 8 \), \( p(2) = 0 \). Name one factor of \( f(x) \).

LESSON 18-2 PRACTICE

13. Determine all the possible rational roots of \( f(x) = x^3 - 5x^2 - 17x + 21 \).

14. Use the Remainder Theorem to determine which of the possible rational roots for the function in Item 13 are zeros of the function.

15. Use the information from Item 14 to graph the function in Item 13.

16. Determine the possible number of positive and negative real roots for \( h(x) = 2x^3 + x^2 - 5x + 2 \).

17. Model with mathematics. Graph \( h(x) = 2x^3 + x^2 - 5x + 2 \).

18. Reason quantitatively. Use the Rational Root Theorem to write a fourth-degree polynomial function that has possible rational roots of \( ±\frac{1}{4}, ±\frac{7}{4}, ±1, ±7 \). Then use Descartes' Rule of Signs to modify your answer to ensure that none of the actual zeros are positive rational numbers.